Consequences of duality

Complementary slackness
Let

$$(P) \min \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$$
$$(D) \max \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \leq \mathbf{c}$$

Theorem Let $\mathbf{x}$ and $\mathbf{y}$ be feasible for (P) and (D). Vectors $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions iff for all i

- $x_i > 0 \Rightarrow \mathbf{y}^T\mathbf{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\mathbf{a}_i < c_i$

where $\mathbf{a}_i$ is ith column of A.

1: Prove the theorem. Consider $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} =?$

Solution: conditions $\Rightarrow$ optimality:

$$\begin{aligned} \left(\mathbf{y}^T A - \mathbf{c}^T\right)\mathbf{x} &= 0 \\ \mathbf{y}^T A\mathbf{x} - \mathbf{c}^T\mathbf{x} &= 0 \\ \mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} &= 0 \\ \mathbf{y}^T\mathbf{b} &= \mathbf{c}^T\mathbf{x} \end{aligned}$$

conditions $\Leftarrow$ optimality: Reverse the computation and observe that $\left(\mathbf{y}^T A - \mathbf{c}^T\right) \leq 0$ and $\mathbf{x} \geq 0$. Hence every coordinate must be 0, which are the constraints.

Example: Diet problem. Suppose in optimal solution $\mathbf{a_i x} > b_i$. Then $y_j = 0$. In an optimal solution we get more of some nutrient than we need, so the cost in the dual is zero.

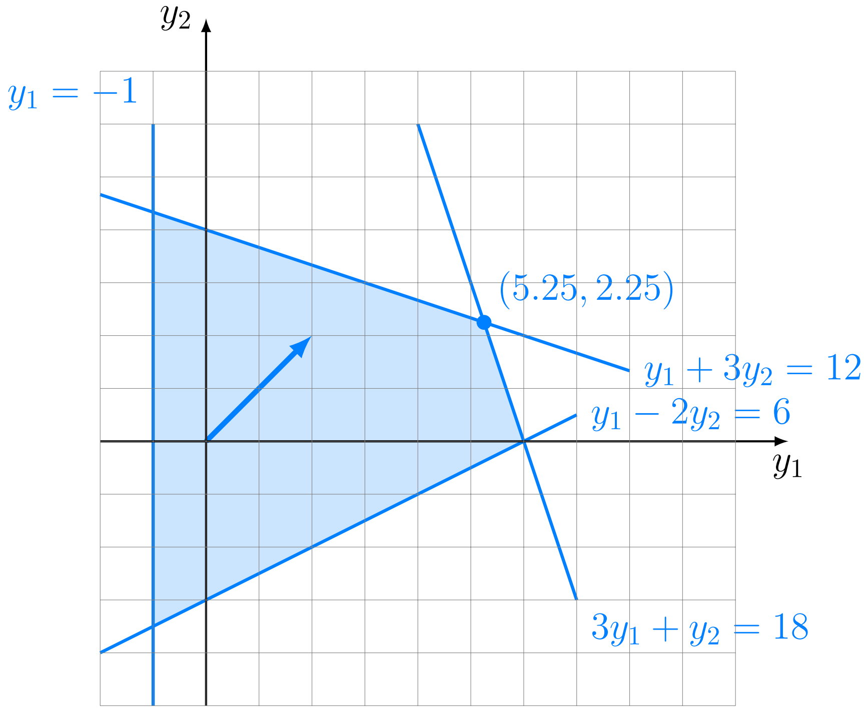
Geometric interpretation and solutions with complementary slackness

$$(P)\begin{cases} \min & 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ \text{s.t} & 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ & x_1 + 3x_2 + 0x_3 - 2x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

2: Write the dual and solve it using graphical method (draw half-spaces). Then reconstruct the solution of (P).

Solution:

$$(D)\begin{cases} \max & 2y_1 + 2y_2 \\ \text{s.t} & 3y_1 + y_2 \leq 18 \\ & y_1 + 3y_2 \leq 12 \\ & -2y_1 + 0y_2 \leq 2 \\ & y_1 - 2y_2 \leq 6 \end{cases}$$



Solution will be intersection of $3y_1 + y_2 = 18$ and $y_1 + 3y_2 = 12$, which gives $y_1 = 5.25, y_2 = 2.25$ and objective value is 15.

Complementary slackness implies that $x_3 = x_4 = 0$. This gives us

$$(P')\begin{cases} \min & 18x_1 + 12x_2 \\ \text{s.t} & 3x_1 + x_2 = 2 \\ & x_1 + 3x_2 = 2 \\ & x_1, x_2 \geq 0 \end{cases}$$

Notice that original matrix

$$A = \begin{pmatrix} 3 & 1 & -2 & 1 \\ 1 & 3 & 0 & -2 \end{pmatrix}$$

can be written as $A = (B|\text{ trash})$ and the solution can be computed by solving $B^{-1}\mathbf{b}$. The solution is $x_1 = x_2 = \frac{1}{2}$. The objective value is also 15.

Sensitivity

Let

$$(P) \min \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$$
$$(D) \max \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \leq \mathbf{c}$$

How does the solution change if $\mathbf{b}$ changes?

Which of the constraints are important and which are not?

Consider optimal solution $\mathbf{x}^\star = (\mathbf{x}_B, 0)$. Then $A = (B|\text{trash})$. Submatrix B is called the *base* of the solution. Note $\mathbf{x}_B = B^{-1}\mathbf{b}$.

$$\mathbf{c}^T\mathbf{x}^\star = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$$

Hence $(\mathbf{y}^\star)^T = \mathbf{c}_B^T B^{-1}$.

Suppose $\mathbf{b} \to (\mathbf{b} + \Delta\mathbf{b})$. If $\Delta\mathbf{b}$ small, base B is still the same. (see example) Then the new optimal solution is

$$B^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \mathbf{x}_B + \Delta\mathbf{x}_B.$$

3: What will be the change of the value of the objective function? (denoted by Δz)

Solution:

$$\Delta z = \mathbf{c}_B^T \cdot \Delta\mathbf{x} = \mathbf{c}_B^T \cdot B^{-1}(\Delta\mathbf{b}) = (\mathbf{y}^\star)^T(\Delta\mathbf{b})$$

So if $\mathbf{b}$ is changed by $\Delta\mathbf{b}$, then the value of objective function is changed by $(\mathbf{y}^\star)^T(\Delta\mathbf{b})$.

$\mathbf{y}^\star$ gives sensitivity of the solution.

Let

$$(P) \min \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0$$
$$(D) \max \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0$$

Then complementary slackness gives

- $x_i > 0 \Rightarrow \mathbf{y}^T\mathbf{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\mathbf{a}_i < c_i$
- $y_i > 0 \Rightarrow \mathbf{a}^i\mathbf{x} = b_i$
- $y_i = 0 \Leftarrow \mathbf{a}^i\mathbf{x} > b_i$,

where $\mathbf{a}^i$ is ith row of A and where $\mathbf{a}_i$ is ith column of A.